# Fast Algorithms for High-Order Sparse Linear Prediction with Applications to Speech Processing

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# Real-time optimization in signal processing



- Hard real-time (a solution is required at a certain time).
- General optimization in signal processing: as fast as possible.
- Current well-known methods: NLMS, RLS, LPC analysis/synthesis, Kalman filtering, Viterbi (decoding)....
- Real-time optimization for more complicated problems:
  - More complicated constraints
  - General convex problems or possible non-convex problems.
  - Non-smooth problems.

# Principles of linear prediction



- A stationary set of samples of speech $x[t]$, for $t = 1, \ldots, T$, are written as a linear combination of $N$ past samples

$$x[t] = \sum_{n=1}^{N} \alpha_n x[t-n] + r[t], \tag{1}$$

- $\{\alpha_n\}$ are the prediction coefficients and $r[t]$ is the prediction error.
- Matrix formulation (certain boundary conditions)

$$x = X\alpha + r \tag{2}$$

- Find the prediction coefficients via

$$\underset{\alpha}{\text{minimize}} \quad \|x - X\alpha\|_p^p \tag{3}$$

# Conventional linear prediction



- Select $p = 2$

$$\underset{\alpha}{\text{minimize}} \quad \|x - X\alpha\|_2^2 \tag{4}$$

- Solution satisfying the normal equation

$$X^T X\alpha = X^T x \tag{5}$$

- The autocorrelation matrix $R = X^T X$, is Toeplitz and with the special right-hand side, $X^T x$ the system can be solved using the Levinson–Durbin algorithm in $\mathcal{O}(N^2)$.

# Long-term prediction



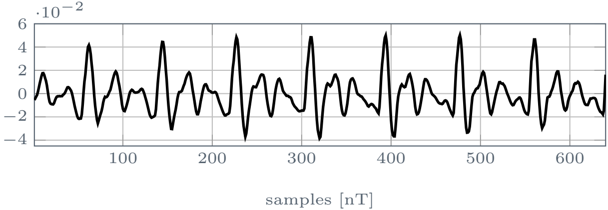
- Generally, linear prediction models only short-term redundancies of speech, thus is often used in combination with a single-tap or multi-tap long-term predictor[1]. The speech model for the long-term predictor is

$$d[t] = \sum_{k=0}^{K} \phi_k d[t - T_\mathrm{p} - k] + r[t], \qquad (6)$$

- $\{\phi_k\}$ are the (long term) prediction coefficients and $r[t]$ is the prediction error, and pitch period $T_\mathrm{p}$ [in samples].

[1] P. Kabal and R.P. Ramachandran. "Joint optimization of linear predictors in speech". In: *Acoustics, Speech and Signal Processing, IEEE Transactions on* 37.5 (1989), pp. 642–650. ISSN: 0096-3518.

- The combination of short term and long prediction filter can be seen as a sparse high order filter:



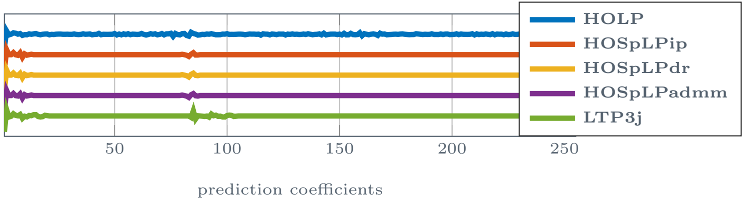




Figure : A 640 samples segment of the voiced speech (vowel /a/ uttered by a female speaker) and some predictors.

# High-order sparse linear prediction



- Imposing sparsity via the 1-norm convex relaxation:

$$\underset{\alpha}{\text{minimize}} \quad \|x - X\alpha\|_2^2 + \gamma\|\alpha\|_1 \,. \tag{7}$$

- However, when imposing sparsity on both the residual vector and high-order predictor, gains can been obtained both in terms of modeling and coding performance[2]

$$\underset{\alpha}{\text{minimize}} \quad \|x - X\alpha\|_1 + \gamma\|\alpha\|_1 \,. \tag{8}$$

- In general, check out[3].

[2] D. Giacobello et al. "Speech coding based on sparse linear prediction". In: *Proc. of the European Signal Processing Conference (EUSIPCO)*. 2009, pp. 2524–2528.

[3] Daniele Giacobello et al. "Sparse linear prediction and its applications to speech processing". In: *Audio, Speech, and Language Processing, IEEE Transactions on* 20.5 (2012), pp. 1644–1657.

# Solving the sparse linear prediction problem



- The objective:

$$f(\alpha) = \|x - X\alpha\|_1 + \gamma\|\alpha\|_1 \tag{9}$$

- is convex but not differentiable, neither is any of the terms $\rightarrow$ proximal gradient methods are not applicable[4][5][6].

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[4] Yu. Nesterov. *Gradient methods for minimizing composite objective function.* Université catholique de Louvain, Center for Operations Research and Econometrics (CORE). No 2007076, CORE Discussion Papers. 2007.

[5] A. Beck and M. Teboulle. "A fast iterative shrinkage-thresholding algorithm for linear inverse problems". In: *SIAM Journal of Imaging Sciences* 2 (1 2009), pp. 183–202.

[6] S. J. Wright, R.D. Nowak, and M. A. T. Figueiredo. "Sparse reconstruction by separable approximation". In: *Signal Processing, IEEE Transaction on* 57 (2009), pp. 2479–2493.

# Solving the sparse linear prediction problem I



- The problem can be solved as a general linear programming problem using interior-point methods[7][8].
- Work-load concentrated on solving linear systems with coefficient matrix

$$C = X^T D_1 X + D_2, \quad D_1, D_2 \text{ diagonal, change at each iteration} \tag{10}$$

- With $X \in \mathbb{R}^{260\times 100}$: solved in $\simeq 10\,\mathrm{ms}$ on a standard laptop computer using C++ and MKL BLAS.

[7] Ghasem Alipoor and Mohammad Hasan Savoji. "Wide-band speech coding based on bandwidth extension and sparse linear prediction". In: *Telecommunications and Signal Processing (TSP), 2012 35th International Conference on*. IEEE. 2012, pp. 454–459.

[8] T.L. Jensen et al. "Real-time implementations of sparse linear prediction for speech processing". In: *Acoustics, Speech and Signal Processing (ICASSP), 2013 IEEE International Conference on*. IEEE. 2013, pp. 8184–8188.

# Solving the sparse linear prediction problem II



- Can it be done faster/more efficient?
- Is it possible to exploit the structure of $X$ and $R = X^T X$?
- Is the high accuracy of the IP methods necessary?

# Solving the sparse linear prediction problem III



- Investigate Douglas-Rachford and Alternating Directional Method of Multipliers (ADMM)
- Can be understood as dual methods of each other.
- Long history but have recently gained interested, also in signal processing[9][10].

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[9] M.V. Afonso, J.M. Bioucas-Dias, and M.A.T. Figueiredo. "Fast Image Recovery Using Variable Splitting and Constrained Optimization". In: *Image Processing, IEEE Transactions on* 19.9 (2010), pp. 2345–2356. ISSN: 1057-7149.

[10] J. Yang and Y. Zhang. "Alternating Direction Algorithms for $\ell_1$-Problems in Compressive Sensing". In: *SIAM Journal of Scientific Computing* 33.1 (2011), pp. 250–278.

# Douglas-Rachford I



- Write the problem as

$$\underset{\alpha}{\text{minimize}} \quad f_1(\alpha) + f_2(X\alpha) \tag{11}$$

- $f_1(u) = \gamma\|u\|_1$ and $f_2(u) = \|x - u\|_1$.
- Let $h(u_1, u_2) = f_1(u_1) + f_2(u_2)$, then the problem can be written as

$$\begin{aligned} \underset{u_1,u_2}{\text{minimize}} \quad & h(u_1, u_2) \\ \text{subject to} \quad & u_2 = Xu_1 \, . \end{aligned} \tag{12}$$

# Douglas-Rachford I



- One form of the Douglas-Rachford algorithm is then

$$u^{(k+1)} = \mathbf{prox}_{th}(z^{(k)}) \tag{13}$$

$$y^{(k+1)} = \mathcal{P}_{\mathbb{Q}}(2u^{(k+1)} - z^{(k)}) \tag{14}$$

$$z^{(k+1)} = z^{(k)} + \eta(y^{(k+1)} - u^{(k+1)}) \tag{15}$$

- Relaxation parameter $\rho \in (0, 2)$, step-size parameter $t > 0$, set $\mathbb{Q} = \{ [u_1, u_2]^T \mid u_2 = Xu_1 \}$
- For smooth and strongly convex problems there are optimal choices for $t, \rho$. For non-smooth it is more heuristics[11].
- In this form, also known as Spingarns method[12].

[11] P. Patrinos, L. Stella, and A. Bemporad. *Douglas-Rachford splitting: complexity estimates and accelerated variants.* Proc. 53rd IEEE Conference on Decision and Control (CDC). 2014.

[12] J.E. Spingarn. "Applications of the method of partial inverses to convex programming: Decomposition". In: *Mathematical Programming* 32.2 (1985), pp. 199–223.

# Douglas-Rachford II



- Step one and three is simply soft-thresholding and level 1 BLAS.
- The projection in step 2 is

$$\mathcal{P}_{\mathbb{Q}}(v) = \begin{bmatrix} I \\ X \end{bmatrix} (I + X^T X)^{-1} (v_1 + X^T v_2)\,. \qquad (16)$$

- To compute (16) we need to solve a linear system of equations with (constant) coefficient matrix $I + X^T X$ and varying right-hand sides $(v_1 + X^T v_2)$.
- Recall $R = X^T X$ (symmetric and Toeplitz).

# ADMM I



- Reformulate as a basis pursuit problem

$$\begin{aligned} \underset{\tilde{z}}{\text{minimize}} \quad & \|\tilde{z}\|_1 \\ \text{subject to} \quad & \tilde{X}\tilde{z} = \tilde{x} \end{aligned} \tag{17}$$

- with

$$\tilde{X} = \begin{bmatrix} X & \gamma I \end{bmatrix} \tag{18}$$

$$\tilde{x} = \gamma x \,. \tag{19}$$

# ADMM II



- This problem formulation readily brings us to an ADMM algorithm defined by the iterations:

$$\tilde{z}^{(k+1)} = \mathcal{P}_{\mathbb{U}}(\tilde{y}^{(k)} - \tilde{u}^{(k)}) \tag{20}$$

$$\tilde{y}^{(k+1)} = \mathcal{S}_{1/\rho}(\tilde{z}^{(k+1)} + \tilde{u}^{(k)}) \tag{21}$$

$$\tilde{u}^{(k+1)} = \tilde{u}^{(k)} + \tilde{z}^{(k+1)} - \tilde{y}^{(k+1)}\,. \tag{22}$$

- where $\mathbb{U} = \{\tilde{z} \in \mathbb{R}^{m+n} \,|\, \tilde{X}\tilde{z} = \tilde{x}\}$

# ADMM II



- We find it instructive to write the algorithm in the form:

$$\alpha^{(k+1)} = \alpha_{\gamma,2} - \begin{bmatrix} -\gamma I \\ X \end{bmatrix}^{+} \left(y^{(k)} - u^{(k)}\right) \tag{23}$$

$$e^{(k+1)} = x - X\alpha^{(k+1)} \tag{24}$$

$$y^{(k+1)} = \mathcal{S}_{1/\rho}\left(\begin{bmatrix} \gamma\alpha^{(k+1)} \\ e^{(k+1)} \end{bmatrix} + u^{(k)}\right) \tag{25}$$

$$u^{(k+1)} = u^{(k)} + \begin{bmatrix} \gamma\alpha^{(k+1)} \\ e^{(k+1)} \end{bmatrix} - y^{(k+1)}. \tag{26}$$

- where $\alpha_{2,\gamma} = (X^T X + \gamma I)^{-1} X^T x$ and $(\cdot)^+$ denotes the Moore-Penrose pseudo-inverse.
- Note that with $\tilde{y}^{(0)} - \tilde{u}^{(0)} = 0$, we have $\alpha^{(1)} = \alpha_{\gamma,2}$, and the ADMM algorithm can then be interpreted as iterative "sparsification" of the $\ell_2$-regularized "classical" linear prediction solution.

# Solving positive-definite symmetric Toeplitz systems I



- Fast algorithms[13] $\rightarrow \mathcal{O}(N^2)$.
- Superfast algorithms[14] $\rightarrow \mathcal{O}(N \log^2 N)$ subsequent solves: $\mathcal{O}(N \log N)$.
- "Intermediate"[15] $\rightarrow \mathcal{O}(N^2)$ subsequent solves: $\mathcal{O}(N \log N)$.
- Break-even point in the number of operations at approximately $N = 256$ for $N$ as a radix 2 number. We will use $N = 250$, so go for the intermediate.

[13] N. Levinson. "The Weiner RMS Error Criterion in Filter Design and Prediction". In: *Journal of Mathematics and Physics* 25 (1947), pp. 261–278.

[14] R.R. Bitmead and B.D.O Anderson. "Asymptotically fast solution of Toeplitz and related systems of linear equations". In: *Linear Algebra and its Applications* 34 (1980), pp. 103–116; G.S. Ammar and W.B Gragg. "Superfast solution of real positive definite Toeplitz systems". In: *SIAM Journal on Matrix Analysis and Applications* 9.1 (1988), pp. 61–76.

[15] J. R. Jain. "An efficient algorithm for a large Toeplitz set of linear equations". In: *Acoustics, Speech and Signal Processing, IEEE Transaction on* 27.6 (1979).

# Solving positive-definite symmetric Toeplitz systems I



The inverse of a Toeplitz matrix can be described by the Gohberg-Semencul formula

$$\delta_N T^{-1} = T_1 T_1^T - T_0^T T_0 \tag{27}$$

where

$$T_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \rho_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \rho_{N-1} & \cdots & \rho_0 & 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \rho_{N-1} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \rho_0 & \cdots & \rho_{N-1} & 1 \end{bmatrix}. \tag{28}$$

The variables $\delta_N$ and $\rho_0, \ldots, \rho_{N-1}$ is a by-product of the Durbin algorithm (or Szegő recursions).

# Solving positive-definite symmetric Toeplitz systems II



- The solution to the system $Tx = b$ is then given by

$$x = T^{-1}b = \frac{1}{\delta_N}\left(T_1 T_1^T b - T_0^T T_0 b\right) . \qquad (29)$$

- Evaluation of matrix-vector products with $T_0, T_1$ is possible via FFTs/IFFTs.

# Timings

Results in ms on a standard desktop, single sentence, 131 frames of 20ms.

| Methods | Timings |
|---|---|
| **CVX+SeDuMi** | 1327.29/2467.80/3619.74 |
| **Mosek** | 145.54/224.71/307.60 |
| **Cprimal** | 55.24/92.70/180.46 |
| **Cprimal(s/d)** | 33.59/63.66/112.09 |
| **DR-L** | 0.65/6.62/10.11 |
| **DR-GS** | 0.61/2.28/3.26 |
| **ADMM-L** | 0.65/2.99/5.14 |
| **ADMM-GS** | 0.61/1.29/1.92 |

Table : Timing in milliseconds. Format: min/average/max. The settings are $T = 320$, $N = 250$ ($M = 570$).

# Convergence behaviour example

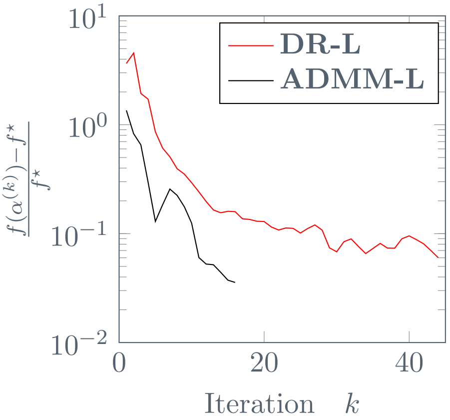


Figure : The endpoints of the graphs illustrates where the stopping criteria has become active and stopped the iterative algorithm.

# Convergence behaviour



- The splitting methods solved the problem to a low accuracy. Define the metrics

$$m_{\mathbf{DR}} = \frac{f_{\mathbf{DR}} - f^\star}{f^\star}, \qquad m_{\mathbf{ADMM}} = \frac{f_{\mathbf{ADMM}} - f^\star}{f^\star} \tag{30}$$

- On average $m_{\mathbf{DR}}$ and $m_{\mathbf{ADMM}}$ is 0.14 and 0.12, respectively.
- ADMM uses 13.5 iterations on average, while the DR based algorithms uses 35.3 iterations on average.
- Sub-optimal solutions can still provide exactly sparse solutions due to the soft-thresholding function.
- Do we only need a sparse and “small” solution with “small” residual?

# Prediction gain



| METHOD | $N$ = 320 | $N$ = 640 |
|---|---|---|
| **LTP1** | 17.3±0.8 | 14.2±1.0 |
| **LTP3** | 22.3±0.8 | 19.9±0.9 |
| **LTP3j** | 24.2±0.6 | 22.6±0.8 |
| **HOLP** | 32.4±0.6 | 31.3±0.7 |
| **HOSpLPip** | 28.6±1.1 | 27.8±1.4 |
| **HOSpLPdr** | 28.5±1.4 | 27.6±1.6 |
| **HOSpLPadmm** | 28.3±1.7 | 27.2±1.6 |

Table : Average prediction gains [dB] for segments of different length $N$, TIMIT database, only voiced speech frames. A 95% confidence interval is shown. The number of nonzero elements, $\mathbf{card}(\cdot)$, is shown for comparison. Fixed $\gamma = 0.12$.

# AR Interpolation I



- A segment of known and unknown samples

$$x = Kx_{\mathrm{k}} + Ux_{\mathrm{u}}, \tag{31}$$

- where $U$ and $K$ are $T \times T$ "rearrangements"
- If the AR coefficients are known, the residual is

$$r = A(Kx_{\mathrm{k}} + Ux_{\mathrm{u}}) \tag{32}$$

- with $A$ the so-called analysis matrix obtained from $\alpha$.
- The least-squares solution is

$$x_{\mathrm{u}} = -\left(A_{\mathrm{u}}^T A_{\mathrm{u}}\right)^{-1} A_{\mathrm{u}}^T A_{\mathrm{k}} x_{\mathrm{k}} \tag{33}$$

with $A_{\mathrm{u}} = AU$ and $A_{\mathrm{k}} = AK$.

# AR Interpolation II



| METHOD | $T_{\mathrm{GAP}}$ 4 | 6 | 8 | 10 | 20 |
|---|---|---|---|---|---|
| **sLP** | 3.92±0.09 | 3.15±0.15 | 2.96±0.16 | 2.30±0.18 | 1.71±0.22 |
| **LTP1** | 4.13±0.07 | 3.44±0.14 | 3.17±0.12 | 2.71±0.09 | 2.45±0.13 |
| **LTP3** | 4.17±0.07 | 3.53±0.09 | 3.22±0.13 | 2.92±0.12 | 2.63±0.09 |
| **LTPj** | 4.12±0.05 | 3.63±0.12 | 3.31±0.12 | 3.00±0.11 | 2.75±0.16 |
| **HOLP** | 4.27±0.04 | 3.55±0.06 | 3.34±0.08 | 2.91±0.09 | 2.61±0.11 |
| **HOSpLPip** | 4.34±0.03 | 3.75±0.05 | 3.56±0.08 | 3.27±0.09 | 3.12±0.15 |
| **HOSpLPdr** | 4.34±0.02 | 3.74±0.08 | 3.55±0.07 | 3.27±0.11 | 3.12±0.12 |
| **HOSpLPadmm** | 4.31±0.04 | 3.69±0.07 | 3.54±0.07 | 3.24±0.08 | 3.11±0.11 |

- Use $\alpha$ and $x_{\mathrm{k}}$ from previously known frame of size 40 ms.
- 1000 sentences from the TIMIT database (both voiced and unvoiced).
- $T_{\mathrm{GAP}}$ is the length of the unknown vector measured in ms.
- Average MOS for speech reconstruction with different gap size losses. A 95% confidence interval is shown.

# Conclusion



- Propose fast algorithms for sparse linear prediction.
- Usage of $\mathcal{O}(N \log N)$ algorithms for repeated solve of positive definite symmetric Toeplitz systems.
- The low accuracy solution provided by the fast algorithms allows to be implemented in real-time systems, particularly in wideband speech processing.
- Experimental evidence obtained through perceptually objective measures shows that the low accuracy solution performs as good as the high accuracy solution when applied in a autoregressive model-based speech reconstruction framework.